An introduction to Dynamic Linear Models

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Dynamic linear models (DLMs)

- DLMs are another form of MARSS model
- But, their underlying structure is different from others we've examined
- General idea is to allow for "evolution" of parameters over time
- Can be univariate (y_t) or multivariate ($\mathbf{y}_t$) in the response

References for DLMs

Petris G, Petrone S, Campagnoli P. 2009. Dynamic Linear Models with R. Springer, New York

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Lamon EC, Carpenter SR, Stow CA. 1998. Forecasting PCB concentrations in Lake Michigan salmonids: a dynamic linear model approach. *Ecol. Appl*. 8: 659-668

Scheuerell MD, Williams JG. 2005. Forecasting climate-induced changes in the survival of Snake River spring/summer Chinook salmon. *Fish. Ocean*. 14: 448-457

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Simple linear regression

- Let's begin with *static* (simple) linear regression with Gaussian errors

- The idea is that the i^{th} observation is function of an intercept and explanatory variable(s)

$$y_i = \alpha + \beta F_i + v_i \qquad v_i \sim \mathrm{N}\left(0, \sigma^2\right)$$

- Importantly, the index *i* has no explicit/implicit meaning—shuffling (y_i, F_i) pairs has no effect on parameter estimation or interpretation

Linear regression in matrix form

- We can write the model in matrix notation

$$y_i = \alpha + \beta F_i + v_i$$

$$y_i = \begin{pmatrix} 1 & F_i \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + v_i$$

$$y_i = \mathbf{F}_i^{\mathrm{T}} \boldsymbol{\theta} + v_i$$

where $\mathbf{F}_i^{\mathrm{T}} = \begin{pmatrix} 1 & F_i \end{pmatrix}$ & $\boldsymbol{\theta} = \begin{pmatrix} \alpha & \beta \end{pmatrix}^{\mathrm{T}}$

Dynamic linear model*

- In a *dynamic* linear model, the regression parameters change over time, so we write

$$y_i = \mathbf{F}_i^{\mathrm{T}} \boldsymbol{\theta} + v_i \qquad \textit{(static)}$$

as

$$y_t = \mathbf{F}_t^{\mathrm{T}} \boldsymbol{\theta}_t + v_t \qquad \textit{(dynamic)}$$

1) Subscript *t* explicitly acknowledges implicit info in time ordering of data

2) Relationship between *y* and **F** is unique at every *t*

*univariate in the response

Constraining a DLM

- Close examination of the DLM reveals an apparent complication for parameter estimation

$$y_t = \mathbf{F}_t^{\mathrm{T}} \boldsymbol{\theta}_t + v_t$$

- With only 1 obs at each *t*, we could only hope to estimate 1 parameter (with no uncertainty)!

- To address this, we will constrain the regression parameters to be dependent from *t* to *t*+1

$$\boldsymbol{\theta}_t = \mathbf{G}_t \boldsymbol{\theta}_{t-1} + \mathbf{w}_t \qquad \mathbf{w}_t \sim \mathrm{MVN}(0, \mathbf{Q})$$

In practice, we will typically make **G** time invariant
& often set **G** = **I**

DLM in matrix form*

State or "evolution" equation

$$\boldsymbol{\theta}_t = \mathbf{G}_t \boldsymbol{\theta}_{t-1} + \mathbf{w}_t \qquad \mathbf{w}_t \sim \text{MVN}(0, \mathbf{Q})$$

Determines how parameters change over time

Observation equation

$$y_t = \mathbf{F}_t^{\mathrm{T}} \boldsymbol{\theta}_t + v_t \qquad v_t \sim \text{N}(0, r)$$

Relates explanatory variable(s) to the observation

*univariate in the response

DLM in MARSS notation

State or "evolution" equation

DLM: $\boldsymbol{\theta}_t = \mathbf{G}_t \boldsymbol{\theta}_{t-1} + \mathbf{w}_t \qquad \mathbf{w}_t \sim \mathrm{MVN}(0, \mathbf{Q})$

MARSS: $\mathbf{x}_t = \mathbf{B}_t \mathbf{x}_{t-1} + \mathbf{w}_t \qquad \mathbf{w}_t \sim \mathrm{MVN}(0, \mathbf{Q})$

Observation equation

DLM: $y_t = \mathbf{F}_t^{\mathrm{T}} \boldsymbol{\theta}_t + v_t \qquad v_t \sim \mathrm{N}(0, r)$

MARSS: $y_t = \mathbf{Z}_t \mathbf{x}_t + v_t \qquad v_t \sim \mathrm{N}(0, r)$

Contrast in covariate effects

Note: DLMs include covariate effects in obs eqn much differently than other forms of MARSS models

DLM: $y_t = \mathbf{F}_t^{\mathrm{T}} \boldsymbol{\theta}_t + v_t \qquad v_t \sim \mathrm{N}(0, r)$

DLM in MARSS: $y_t = \mathbf{Z}_t \mathbf{x}_t + v_t \qquad v_t \sim \mathrm{N}(0, r)$

Other MARSS: $y_t = \mathbf{Z}_t \mathbf{x}_t + \mathbf{D}\mathbf{d}_t + v_t \qquad v_t \sim \mathrm{N}(0, r)$

Different forms of DLMs

The univariate regression model is just one example of a DLM—other forms include:

- Stochastic “level” (intercept)
- Stochastic “growth” (trend, bias)
- Seasonal effects (fixed, harmonic)

The most simple univariate DLM

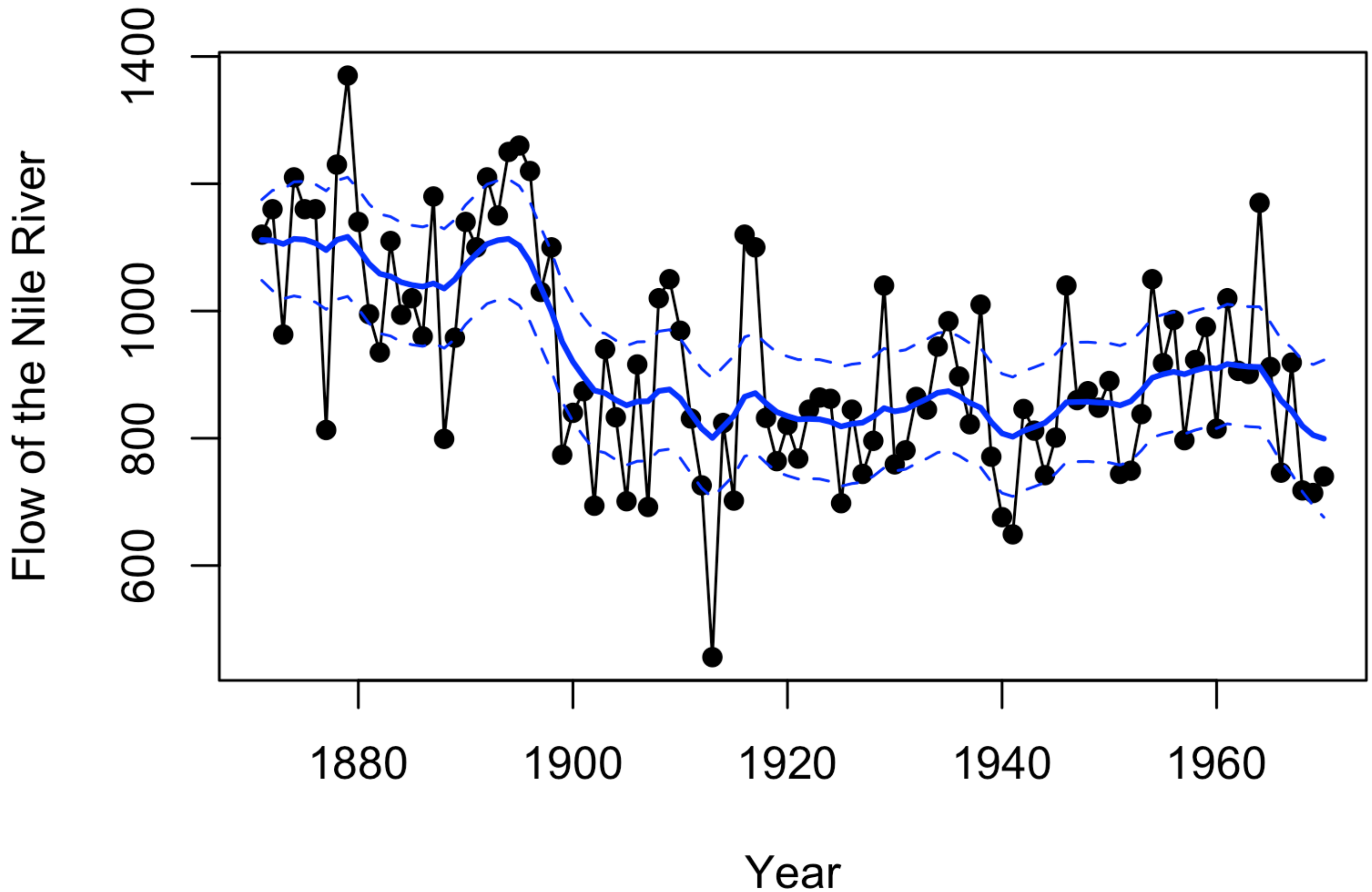
Stochastic "level" (intercept-only)

DLM
$$\begin{cases} \alpha_t = \alpha_{t-1} + w_t & w_t \sim \mathrm{N}(0,q) \\ y_t = \alpha_t + v_t & v_t \sim \mathrm{N}(0,r) \end{cases}$$

A random walk with observation error

MARSS
$$\begin{cases} x_t = x_{t-1} + w_t & w_t \sim \mathrm{N}(0,q) \\ y_t = x_t + v_t & v_t \sim \mathrm{N}(0,r) \end{cases}$$

The most simple univariate DLM



The most simple multivariate DLM

Multiple observations of a single random walk

$$x_t = x_{t-1} + w_t \qquad w_t \sim \mathrm{N}(0, q)$$

$$\mathbf{y}_t = \mathbf{Z} x_t + \mathbf{v}_t \qquad \mathbf{v}_t \sim \mathrm{MVN}(0, \mathbf{R})$$

$$\mathbf{Z} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Another simple multivariate DLM

Multiple observations of multiple random walks

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{w}_t \qquad \mathbf{w}_t \sim \text{MVN}(0, \mathbf{Q})$$

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{v}_t \qquad \mathbf{v}_t \sim \text{MVN}(0, \mathbf{R})$$

$$\mathbf{Z} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} = \mathbf{I}$$

Univariate DLM for level & growth

Stochastic "level" with deterministic "growth"

DLM

$$\begin{cases} \alpha_t = \alpha_{t-1} + \gamma + w_t & w_t \sim \mathrm{N}(0,q) \\ y_t = \alpha_t + v_t & v_t \sim \mathrm{N}(0,r) \end{cases}$$

Random walk with drift

MARSS

$$\begin{cases} x_t = x_{t-1} + u + w_t & w_t \sim \mathrm{N}(0,q) \\ y_t = x_t + v_t & v_t \sim \mathrm{N}(0,r) \end{cases}$$

Univariate DLM for level & growth

Stochastic "level" with stochastic "growth"

Level: $\alpha_t = \alpha_{t-1} + \gamma_{t-1} + w_t^{(1)}$ $\quad w_t^{(1)} \sim \mathrm{N}(0, q_1)$

Growth: $\gamma_t = \gamma_{t-1} + w_t^{(2)}$ $\quad w_t^{(2)} \sim \mathrm{N}(0, q_2)$

Level: $\alpha_t = 1\alpha_{t-1} + 1\gamma_{t-1} + w_t^{(1)}$ $\quad w_t^{(1)} \sim \mathrm{N}(0, q_1)$

Growth: $\gamma_t = 0\alpha_{t-1} + 1\gamma_{t-1} + w_t^{(2)}$ $\quad w_t^{(2)} \sim \mathrm{N}(0, q_2)$

Level:
Growth:

$$\begin{bmatrix} \alpha_t \\ \gamma_t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{t-1} \\ \gamma_{t-1} \end{bmatrix} + \begin{bmatrix} w_t^{(1)} \\ w_t^{(2)} \end{bmatrix}$$

Univariate DLM for level & growth

Stochastic "level" with stochastic "growth"

Level:
Growth:

$$\underbrace{\begin{bmatrix} \alpha_t \\ \gamma_t \end{bmatrix}}_{\boldsymbol{\theta}_t} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix} \alpha_{t-1} \\ \gamma_{t-1} \end{bmatrix}}_{\boldsymbol{\theta}_{t-1}} + \underbrace{\begin{bmatrix} w_t^{(1)} \\ w_t^{(2)} \end{bmatrix}}_{\mathbf{w}_t}$$

DLM: $\boldsymbol{\theta}_t = \mathbf{G}\boldsymbol{\theta}_{t-1} + \mathbf{w}_t$ $\quad \mathbf{w}_t \sim \mathrm{MVN}(0, \mathbf{Q})$

MARSS: $\mathbf{x}_t = \mathbf{B}\mathbf{x}_{t-1} + \mathbf{w}_t$ $\quad \mathbf{Q} = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$

Univariate DLM for level & trend

Observation eqn for stochastic "level" and "growth"

Obs: $y_t = \alpha_t + v_t \qquad v_t \sim \mathrm{N}(0, r)$

$$y_t = 1\alpha_t + 0\gamma_t + v_t$$

Define: $\mathbf{F}_t^{\mathrm{T}} = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad \boldsymbol{\theta}_t = \begin{pmatrix} \alpha_t \\ \gamma_t \end{pmatrix}$

DLM: $y_t = \mathbf{F}_t^{\mathrm{T}} \boldsymbol{\theta}_t + v_t$

MARSS: $y_t = \mathbf{Z}_t \mathbf{x}_t + v_t$

Univariate DLM for regression

Stochastic "intercept" with stochastic "slope"

$$\begin{matrix}\text{Intercept:} \\ \text{Slope:}\end{matrix} \quad \underbrace{\begin{bmatrix} \alpha_t \\ \beta_t \end{bmatrix}}_{\boldsymbol{\theta}_t} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{G}} \underbrace{\begin{bmatrix} \alpha_{t-1} \\ \beta_{t-1} \end{bmatrix}}_{\boldsymbol{\theta}_{t-1}} + \underbrace{\begin{bmatrix} w_t^{(1)} \\ w_t^{(2)} \end{bmatrix}}_{\mathbf{w}_t}$$

DLM: $\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1} + \mathbf{w}_t$

$\mathbf{w}_t \sim \text{MVN}(0, \mathbf{Q})$

MARSS: $\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{w}_t$

$$\mathbf{Q} = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$$

Univariate DLM for regression

Observation eqn for stochastic "intercept" and "slope"

Obs: $y_t = \alpha_t + \beta_t f_t + v_t \qquad v_t \sim \mathrm{N}(0, r)$

Define: $\mathbf{F}_t^{\mathrm{T}} = \begin{pmatrix} 1 & f_t \end{pmatrix} \qquad \boldsymbol{\theta}_t = \begin{pmatrix} \alpha_t \\ \beta_t \end{pmatrix}$

DLM: $y_t = \mathbf{F}_t^{\mathrm{T}} \boldsymbol{\theta}_t + v_t$

MARSS: $y_t = \mathbf{Z}_t \mathbf{x}_t + v_t$

Forecasting with univariate DLM

- DLMs are often used in a forecasting context where we are interested in a prediction at time *t* conditioned on data up through time *t*-1
- Beginning with the distribution of θ at time *t*-1 conditioned on data through time *t*-1:

$$\theta_{t-1} \mid y_{1:t-1} \sim \text{MVN}\left(\pi_{t-1}, \Lambda_{t-1}\right)$$

- Then, the predictive distribution for θ_t given $y_{1:t-1}$ is:

$$\theta_t \mid y_{1:t-1} \sim \text{MVN}\left(\mathbf{G}_t \pi_{t-1}, \mathbf{G}_t \Lambda_{t-1} \mathbf{G}_t^{\text{T}} + \mathbf{Q}\right)$$

- And, the one-step ahead predictive distribution for y_t given $y_{1:t-1}$ is:

$$y_t \mid y_{1:t-1} \sim \text{N}\left(\mathbf{F}_t\left[\mathbf{G}_t \pi_{t-1}\right], \mathbf{F}_t\left[\mathbf{G}_t \Lambda_{t-1} \mathbf{G}_t^{\text{T}} + \mathbf{Q}\right]\mathbf{F}_t^{\text{T}} + \mathbf{R}\right)$$

Forecasting with univariate DLM

Don't worry! MARSS will make this easy for you.

- Beginning with the distribution of θ at time t-1 conditioned on data through time t-1:

$$\theta_{t-1} \mid y_{1:t-1} \sim \text{MVN}\left(\pi_{t-1}, \Lambda_{t-1}\right)$$

- Then, the predictive distribution for θ_t given $y_{1:t-1}$ is:

$$\theta_t \mid y_{1:t-1} \sim \text{MVN}\left(\mathbf{G}_t \pi_{t-1}, \mathbf{G}_t \Lambda_{t-1} \mathbf{G}_t^{\text{T}} + \mathbf{Q}\right)$$

- And, the one-step ahead predictive distribution for y_t given $y_{1:t-1}$ is:

$$y_t \mid y_{1:t-1} \sim \text{N}\left(\mathbf{F}_t \left[\mathbf{G}_t \pi_{t-1}\right], \mathbf{F}_t \left[\mathbf{G}_t \Lambda_{t-1} \mathbf{G}_t^{\text{T}} + \mathbf{Q}\right] \mathbf{F}_t^{\text{T}} + \mathbf{R}\right)$$

Diagnostics for DLMs

- Just as we have seen for other models, diagnostics are an important part of fitting DLMs
- When forecasting, we are often interested in the forecast errors (e_t = observed$_t$ - forecast$_t$)
- In particular, DLMs have the following assumptions:

  1) $e_t \sim \mathrm{N}(0, \sigma)$

  2) $\mathrm{cov}(e_t, e_{t-k}) = 0$

- We can check (1) with a QQ-plot and (2) with an ACF

Multivariate DLM

- Here we will examine multiple responses at once, so we need a multivariate DLM
- First, the obs eqn

$$y_t = \mathbf{F}_t^{\mathrm{T}} \boldsymbol{\theta}_t + v_t \qquad v_t \sim \mathrm{N}(0, r)$$

becomes

$$\mathbf{y}_t = \left(\mathbf{F}_t^{\mathrm{T}} \otimes \mathbf{I}_n\right) \boldsymbol{\theta}_t + \mathbf{v}_t \qquad \mathbf{v}_t \sim \mathrm{MVN}(\mathbf{0}, \mathbf{R})$$

Multivariate DLM – obs eqn

$$\mathbf{y}_t = \left(\mathbf{F}_t^{\mathrm{T}} \otimes \mathbf{I}_n\right)\boldsymbol{\theta}_t + \mathbf{v}_t \qquad \mathbf{v}_t \sim \mathrm{MVN}(\mathbf{0}, \mathbf{R})$$

$$\begin{bmatrix} y_{1,t} \\ \vdots \\ y_{n,t} \end{bmatrix} = \begin{bmatrix} 1 & f_t \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multivariate DLM – obs eqn

$$\mathbf{y}_t = \left(\mathbf{F}_t^{\mathrm{T}} \otimes \mathbf{I}_n\right)\boldsymbol{\theta}_t + \mathbf{v}_t \qquad \mathbf{v}_t \sim \mathrm{MVN}(\mathbf{0}, \mathbf{R})$$

$$\begin{bmatrix} y_{1,t} \\ \vdots \\ y_{n,t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & f_t & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & 1 & 0 & 0 & f_t \end{bmatrix} \begin{bmatrix} \alpha_{1,t} \\ \vdots \\ \alpha_{n,t} \\ \beta_{1,t} \\ \vdots \\ \beta_{n,t} \end{bmatrix} + \begin{bmatrix} v_{1,t} \\ \vdots \\ v_{n,t} \end{bmatrix}$$

Multivariate DLM – obs eqn

$$\mathbf{y}_t = \left(\mathbf{F}_t^{\mathrm{T}} \otimes \mathbf{I}_n\right)\boldsymbol{\theta}_t + \mathbf{v}_t \qquad \mathbf{v}_t \sim \mathrm{MVN}(\mathbf{0}, \underline{\mathbf{R}})$$

$$\mathbf{R} = \underline{\begin{bmatrix} r_1 & \gamma_{21} & \cdots & \gamma_{n1} \\ \gamma_{12} & r_2 & & \gamma_{n2} \\ \vdots & & \ddots & \vdots \\ \gamma_{1n} & \gamma_{2n} & \cdots & r_n \end{bmatrix}} \qquad \mathbf{R} = \underline{\begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{bmatrix}}$$

Multivariate DLM – evolution eqn

- The evolution eqn

$$\boldsymbol{\theta}_t = \mathbf{G}_t \boldsymbol{\theta}_{t-1} + \mathbf{w}_t \qquad \mathbf{w}_t \sim \text{MVN}(\mathbf{0}, \mathbf{Q})$$

becomes

$$\boldsymbol{\theta}_t = (\mathbf{G}_t \otimes \mathbf{I}_n)\boldsymbol{\theta}_{t-1} + \mathbf{w}_t \qquad \mathbf{w}_t \sim \text{MVN}(\mathbf{0}, \mathbf{Q})$$

$$\mathbf{G}_t = \mathbf{I}_2 \Rightarrow \mathbf{G}_t \otimes \mathbf{I}_n = \mathbf{I}_{2n}$$

$$\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1} + \mathbf{w}_t$$

Multivariate DLM – evolution eqn

$$\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1} + \mathbf{w}_t \qquad \mathbf{w}_t \sim \mathrm{MVN}(\mathbf{0},\mathbf{Q})$$

$$\begin{bmatrix} \alpha_{1,t} \\ \vdots \\ \alpha_{n,t} \\ \beta_{1,t} \\ \vdots \\ \beta_{n,t} \end{bmatrix} = \begin{bmatrix} \alpha_{1,t-1} \\ \vdots \\ \alpha_{n,t-1} \\ \beta_{1,t-1} \\ \vdots \\ \beta_{n,t-1} \end{bmatrix} + \begin{bmatrix} w_{1,t}^{(\alpha)} \\ \vdots \\ w_{n,t}^{(\alpha)} \\ w_{1,t}^{(\beta)} \\ \vdots \\ w_{n,t}^{(\beta)} \end{bmatrix}$$

Multivariate DLM – evolution eqn

$$\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1} + \mathbf{w}_t \qquad \mathbf{w}_t \sim \text{MVN}(\mathbf{0}, \underline{\mathbf{Q}})$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}^{(\alpha)} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}^{(\beta)} \end{bmatrix} \qquad \mathbf{Q}^{(\cdot)} = \begin{bmatrix} q_1^{(\cdot)} & c_{21}^{(\cdot)} & \cdots & c_{n1}^{(\cdot)} \\ c_{12}^{(\cdot)} & q_2^{(\cdot)} & & c_{n2}^{(\cdot)} \\ \vdots & & \ddots & \vdots \\ c_{1n}^{(\cdot)} & c_{2n}^{(\cdot)} & \cdots & q_n^{(\cdot)} \end{bmatrix}$$

Multivariate DLM – evolution eqn

$$\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1} + \mathbf{w}_t \qquad \mathbf{w}_t \sim \text{MVN}(\mathbf{0}, \underline{\mathbf{Q}})$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1^{(\alpha)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_k^{(\alpha)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_1^{(\beta)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Q}_k^{(\beta)} \end{bmatrix}$$

For $k < n$ "groups"

Topics for lab

- Fitting univariate DLM regression model with MARSS
- Examining “evolution” of parameters
- Examining model fit
- Model diagnostics